The delayed uncoupled continuous-time random walks do not provide a model for the telegraph equation

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It has been alleged in several papers that the so called delayed continuous-time random walks (DCTRWs) provide a model for the one-dimensional telegraph equation at microscopic level. This conclusion, being widespread now, is strange, since the telegraph equation describes phenomena with finite propagation speed, while the velocity of the motion of particles in the DCTRWs is infinite. In this paper we investigate how accurate are the approximations to the DCTRWs provided by the telegraph equation. We show that the diffusion equation, being the correct limit of the DCTRWs, gives better approximations in L_2 norm to the DCTRWs than the telegraph equation. We conclude therefore that, first, the DCTRWs do not provide any correct microscopic interpretation of the one-dimensional telegraph equation, and second, the kinetic (exact) model of the telegraph equation is different from the model based on the DCTRWs.

I. INTRODUCTION

The continuous-time random walks (CTRWs) [1–4] present sufficiently wide and general class of random walks, describing random motions of particles, when both the waiting (sojourn) time between successive jumps and the jump length (including its direction) are generally coupled random variables. The CTRWs have been widely applied in building up models of anomalous diffusion and transport in physics [5–13] (to name a few) and biology [9, 14, 15], and in economical problems [16].

In their original form [1] the CTRWs describe random walks, or, more precisely, jumps, on lattices when only the waiting time is a random variable. In the general case of the uncoupled CTRWs both waiting time and jump length are independent random variables, this leads to the jump model of the CTRWs. The kinetic model of random walks, often called random flights, originates from the Pearson–Rayleigh random walks [4]. In this model the particle moves with constant velocity for a random time along straight lines between points where it changes randomly the direction of movement. The random flights can be considered both in the framework of the generalized linear transport (kinetic) equation [17] and in the framework of the CTRWs, when the waiting time and jump length are coupled random variables. The latter is the velocity or velocity-jump model of the CTRWs [18].

The jump model of the CTRWs has gained high popularity due to its greater simplicity than the kinetic (velocity-jump) model. However, in the jump model the waiting time can generally be arbitrarily small or/and the jump length can be arbitrary long (Levy flights), and, therefore, a walking particle can move with infinite velocity. Thus, the jump model may, in contrast to the kinetic one, violate the principle of causality.

This is similar to the classic *diffusion paradox* [19–23] related to the conventional diffusion equation, which is widely used for approximate macroscopic description of nonanomalous diffusion and Brownian motion. The difference is that in classic Einstein's model of Brownian motion [24], leading to the diffusion equation, the particle moves with finite velocity. In fact Einstein's model is nothing but the symmetric Bernoulli random walk [25], which can be considered as a degenerate case of the CTRWs in the frameworks of both the jump and kinetic models.

The telegraph equation [19, 20, 22, 23, 26] was proposed as an alternative to the diffusion equation. In contrast to the latter, which is parabolic, the telegraph equation is hyperbolic, providing the finite speed of signal propagation. Two- and three-dimensional telegraph equations meet some formal problems since Green's functions can become negative [27]. Though in the case of thermal conduction this can be fixed by imposing restrictions on the heat flux [23]. Yet the one-dimensional telegraph equation avoids the diffusion paradox and provides better model for nonanomalous diffusion than the one-dimensional diffusion equation.

In Ref. [28] the authors discussed microscopic models of the telegraph equation. Aside from the kinetic model they proposed another model based on uncoupled CTRWs with the waiting time distributed according to the gamma law $(t/\theta^2)\,\mathrm{e}^{-t/\theta}$ and the jump length having the finite second moment. The authors called the CTRWs with the gamma-distributed waiting time delayed CTRWs (DCTRWs). Similar arguments were used in Ref. [29] when deriving the telegraph equation with reaction. In Ref. [30] the telegraph equation with reaction was derived formally from the CTRW with exponentially distributed waiting time, and later, in Ref. [31], — from the CTRW with a general waiting time.

Earlier, the telegraph equation was derived in Ref. [32]

from uncoupled CTRWs with the waiting time having more general distribution than the gamma law. We call these CTRWs for brevity also delayed CTRWs (DC-TRWs).

The conclusions that the DCTRWs simulates the telegraph equation at the microscopic level are strange. Indeed, the telegraph equation describes phenomena with finite propagation speed, while the particle in the DC-TRWs may move with infinite velocity, i. e., the DCTRWs do not reflect the principal peculiarity of the telegraph equation. In spite of this discrepancy, one may believe that the telegraph equation still gives a precise approximation to the DCTRWs.

In this paper we investigate how accurate are the approximations to the DCTRWs provided by the telegraph equation. We show that the diffusion equation, being the correct limit of the DCTRWs, gives better approximations to the DCTRWs than the telegraph equation. One can conclude, therefore, that the DCTRWs cannot be a model for the telegraph equation at microscopic level.

The rest of the paper is organized as follows. In Section II we recall some facts concerning the uncoupled CTRWs. Section III shows how the telegraph equation was derived from the DCTRWs in Ref. [28–32]. The asymptotic behaviour of the DCTRWs is discussed in Section IV. In Section V we compare the telegraph and diffusion approximations to two (discrete- and continuous-space) DCTRWs, completely described in Section A, numerically, while asymptotic comparison of the approximations to the continuous-space DCTRW at long times is performed in Section B. Section VI contains some conclusive remarks.

II. UNCOUPLED CONTINUOUS-TIME RANDOM WALKS

In this section we briefly formulate some facts concerning the CTRWs for further reference.

We adopt the one-dimensional model and consider uncoupled CTRWs, when the waiting time and jump length are independent random variables, i. e., the joint probability density $\varphi(x,t)$ for a particle to jump a distance x after waiting a time t is given by

$$\varphi(x,t) = \psi(t)\lambda(x), \quad x \in \mathbb{R}, \quad t \geq 0,$$

where $\psi(t)$ and $\lambda(x)$ are probability densities for the waiting time and jump length, respectively, with obvious normalizations $\int_0^\infty \psi(t)\,dt = 1$ and $\int_{-\infty}^\infty \lambda(x)\,dx = 1$.

We consider here CTRWs in a continuous space, however, they can be similarly described on a lattice, as it was originally proposed in [1], with obvious modifications.

The CTRW is described by the density $\rho_{\mathrm{RW}}(x,t)$, which is a probability density that the particle is at the point x at the time t, so that $\int_{-\infty}^\infty \rho_{\mathrm{RW}}(x,t)\,dx = 1$. The

density $\rho_{\mathrm{RW}}(x,t)$ is given by the integral equation [10, 33]

$$\rho_{\mathrm{RW}}(x,t) = \Psi(t)\delta(x)$$
$$+ \int_0^t \int_{-\infty}^\infty \varphi(x-x', t-t')\rho_{\mathrm{RW}}(x',t')\,dx'\,dt'$$
$$= \Psi(t)\delta(x)$$
$$+ \int_0^t \psi(t-t')\left[\int_{-\infty}^\infty \lambda(x-x')\rho_{\mathrm{RW}}(x',t')\,dx'\right]dt',$$
(II.1)

where $\delta(\cdot)$ is the Dirac delta function, and

$$\Psi(t) = 1 - \int_0^t \psi(t')\,dt' \equiv \int_t^\infty \psi(t')\,dt', \quad t \geq 0,$$

is the survival probability, i. e., the probability that a particle stays at the same position for the time t. Here one assumes that at the initial moment $t=0$ the particle was at the point $x=0$, i. e., $\rho_{\mathrm{RW}}|_{t=0} = \delta(x)$.

The Fourier–Laplace transform of the equation (II.1), denoted by $\mathcal{FL}\cdot$, implies the Montroll–Weiss formula [1, 3]

$$\mathcal{FL}\rho_{\mathrm{RW}}(k,s) = \frac{\mathcal{L}\Psi(s)}{1 - \mathcal{FL}\varphi(k,s)} = \frac{1 - \mathcal{L}\psi(s)}{s\left[1 - \mathcal{FL}\varphi(k,s)\right]}$$
$$= \frac{1 - \mathcal{L}\psi(s)}{s\left[1 - \mathcal{F}\lambda(k)\mathcal{L}\psi(s)\right]}, \quad (II.2)$$

where

$$\mathcal{F}\lambda(k) = \int_{-\infty}^\infty \lambda(x)\,e^{ikx}\,dx$$

is the Fourier transform of λ,

$$\mathcal{L}\psi(s) = \int_0^\infty \psi(t)\,e^{-st}\,dt$$

is the Laplace transform of ψ.

In the uncoupled CTRWs the density ρ_{RW} can be expressed explicitly (such a CTRW is the random walk subordinated to a renewal process) [34, 35]:

$$\rho_{\mathrm{RW}}(x,t) = \sum_{n=0}^\infty p_n(t)\lambda_n(x)$$
$$\equiv \Psi(t)\delta(x) + \sum_{n=1}^\infty p_n(t)\lambda_n(x) \quad (II.3)$$

($\rho_{\mathrm{RW}}(\cdot,t)$, with fixed t, is actually a probability density of a random sum), where $p_n(t)$ is the probability of n jumps occurring up to the time t, and $\lambda_n(x)$ is the probability density of a distance x from the initial position reached by a particle after n jumps. The probability density $\lambda_n(x)$ is given by

$$\lambda_n(x) = \lambda^{n*}(x),$$

where $*$ means the convolution, λ^{n*} means the n-fold convolution of λ with itself, i. e., $\lambda^{0*}(x) = \delta(x)$, $\lambda^{1*}(x) = \lambda(x)$, and $\lambda^{n*}(x) = (\lambda^{(n-1)*} * \lambda)(x)$. Equivalently,

$$\mathcal{F}\lambda_n(k) = [\mathcal{F}\lambda(k)]^n.$$

The probability $p_n(t)$ is given by

$$p_n(t) = (\Psi * \psi^{n*})(t)$$

(clearly $\psi(t) = 0$ and $\Psi(t) = 0$ for $t < 0$), in particular, $p_0(t) = \Psi(t)$. Equivalently,

$$\mathcal{L}p_n(s) = \mathcal{L}\Psi(s)[\mathcal{L}\psi(s)]^n.$$

This completes the set of auxiliary propositions necessary for further considerations.

III. TELEGRAPH APPROXIMATIONS TO THE DELAYED CONTINUOUS-TIME RANDOM WALKS

Consider a family of probability densities for waiting time t [28, 32, 36]

$$\psi_a(t)$$
$$= \begin{cases} \dfrac{a}{\theta\sqrt{1-a}}\,\mathrm{e}^{-t/\theta}\sinh\left(\dfrac{t}{\theta}\sqrt{1-a}\right), & 0 < a < 1, \\[2ex] \dfrac{t}{\theta^2}\,\mathrm{e}^{-t/\theta}, & a = 1, \end{cases}$$
$$\text{(III.1)}$$

(obviously, $\psi_1(t) = \lim_{a\to 1}\psi_a(t)$). The Laplace transform of the density ψ_a is

$$\mathcal{L}\psi_a(s) = \frac{a}{(\theta s + 1)^2 - (1 - a)}$$
$$\equiv \frac{a}{\theta^2}\frac{1}{(s + 1/\theta)^2 - (1 - a)/\theta^2}.$$

The density ψ_1 was used in Refs. [28, 36] (see also Ref. [37]), and it belongs to the family of the gamma distributions. The density ψ_1 is a probability density of a sum of two independent exponentially distributed random variables with the probability density $(1/\theta)\,\mathrm{e}^{-t/\theta}$, and CTRWs with such a distribution of waiting times were called in Ref. [28] the delayed CTRWs (DCTRWs). Note that the mean waiting time for the exponential distribution is θ, and it is 2θ for the distribution ψ_1. The mean waiting time for the distribution ψ_a is $2\theta/a$, i. e., the mean waiting time tends to infinity as $a \to 0$. Completely understanding the conditional character of this notion we call for brevity the CTRWs with the probability densities ψ_a of the waiting time also the DCTRWs. We denote densities describing the DCTRWs by ρ_{DRW}.

The Montroll–Weiss formula (II.2) with $\mathcal{L}\psi_a$ implies

$$\mathcal{F}\mathcal{L}\rho_{\mathrm{DRW}}(k, s) = \frac{\theta s + 2}{s(\theta s + 2) - a[\mathcal{F}\lambda(k) - 1]/\theta}.$$

The straightforward calculations show that the density ρ_{DRW} is a solution to the equation

$$\frac{\theta}{2}\frac{\partial^2 \rho_{\mathrm{DRW}}(x, t)}{\partial t^2} + \frac{\partial \rho_{\mathrm{DRW}}(x, t)}{\partial t}$$
$$= \frac{a}{2\theta}\left[\int_{-\infty}^{\infty}\lambda(x - x')\rho_{\mathrm{DRW}}(x', t)\,\mathrm{d}x' - \rho_{\mathrm{DRW}}(x, t)\right]$$
$$\text{(III.2)}$$

with the initial value conditions

$$\rho_{\mathrm{DRW}}|_{t=0} = \delta(x), \quad \left.\frac{\partial \rho_{\mathrm{DRW}}}{\partial t}\right|_{t=0} = 0.$$

Consider a random walk on a regular one-dimensional lattice, where the distance between nearest-neighbour points is σ, and the coordinates of the lattice points are $x_l = l\sigma$, $l \in \mathbb{Z}$. Suppose that jumps are made to the nearest-neighbour points with equal probability. This discrete-space random walk is described by the probability density $\lambda_{\mathrm{d}}(x) = [\delta(x - \sigma) + \delta(x + \sigma)]/2$, and the equation (III.2) becomes

$$\frac{\theta}{2}\frac{\partial^2 \rho_{\mathrm{DRW}}(x_l, t)}{\partial t^2} + \frac{\partial \rho_{\mathrm{DRW}}(x_l, t)}{\partial t}$$
$$= \frac{a}{2\theta}\left[\frac{\rho_{\mathrm{DRW}}(x_l - \sigma, t) + \rho_{\mathrm{DRW}}(x_l + \sigma, t)}{2}\right.$$
$$\left. - \rho_{\mathrm{DRW}}(x, t)\right] \quad \text{(III.3)}$$

Assuming that the density ρ_{DRW} is smooth with respect to x, one can expand it in the Taylor series in x. If σ is small enough then we neglect the derivatives of an order higher than two, and obtain the approximation for the density ρ_{DRW} given by the solution of the telegraph equation (TE)

$$\frac{\theta}{2}\frac{\partial^2 \rho_{\mathrm{TE}}}{\partial t^2} + \frac{\partial \rho_{\mathrm{TE}}}{\partial t} - \frac{a\sigma^2}{4\theta}\frac{\partial^2 \rho_{\mathrm{TE}}}{\partial x^2} = 0 \quad \text{(III.4)}$$

with the initial conditions

$$\rho_{\mathrm{TE}}|_{t=0} = \delta(x), \quad \left.\frac{\partial \rho_{\mathrm{TE}}}{\partial t}\right|_{t=0} = 0. \quad \text{(III.5)}$$

The same result is obtained if jumps are made not only to the nearest-neighbour points, the only decisive condition is that the variance of the distribution λ_{d} is to be equal to σ^2.

Similar derivation of the telegraph equation from the discrete-space random walk with the probability density ψ_a for waiting time was performed in Ref. [32], where the authors wrote about "the continuum limit" as $\sigma \to 0$.

Note that both the equations (III.2) and (III.3) describe particles, moving with infinite velocity, while the telegraph equation describes phenomena with finite propagation speed.

Another derivation of the telegraph approximation is given in Ref. [28]. This is similar to the above one, but it

is performed in the Fourier–Laplace space. The derivation, given in Ref. [28] for $a = 1$, is in fact as follows. If the distribution of the jump length is symmetric and has the finite second moment with the variance equal to σ^2, then the Fourier transform of the probability density of the jump length is approximately $\mathcal{F}\lambda(k) \approx 1 - \sigma^2 k^2/2$ for $|\sigma k| \ll 1$. After substitution of this $\mathcal{F}\lambda(k)$ and $\mathcal{L}\psi_a(s)$ into the Montroll–Weiss formula (II.2) one obtains that the Fourier–Laplace transform of the density ρ_{DRW} is approximately equal to

$$\mathcal{F}\mathcal{L}\rho_{\mathrm{DRW}}(k,s) \approx \mathcal{F}\mathcal{L}\rho_{\mathrm{TE}}(k,s) \quad \text{for } |\sigma k| \ll 1$$

with

$$\mathcal{F}\mathcal{L}\rho_{\mathrm{TE}}(k,s) = \frac{\theta s + 2}{s(\theta s + 2) + a\sigma^2 k^2/(2\theta)}. \qquad \text{(III.6)}$$

The inverse Fourier–Laplace transform of $\mathcal{F}\mathcal{L}\rho_{\mathrm{TE}}$ implies that the density ρ_{TE} is a solution to the telegraph equation (III.4) with the initial conditions (III.5). The authors of the paper [28] call the telegraph equation (III.4) "the limit of small jumps (diffusive approximation)" of the CTRW with the probability density ψ_1 (III.1) for waiting time.

The conclusion that the solution to the problem (III.4), (III.5) approximates the DCTRW is strange, since the telegraph equation describes phenomena with finite propagation speed. At the same time, the velocity of the motion of the particle in the DCTRW is infinite, since waiting time can be arbitrarily small. Moreover, the telegraph equation obviously is not "the limit of small jumps" of the DCTRWs.

IV. THE ASYMPTOTIC BEHAVIOUR OF THE DELAYED CONTINUOUS-TIME RANDOM WALKS

The asymptotic behaviour of various CTRWs was studied in Refs. [35, 38, 39]. For convenience we briefly repeat the asymptotic analysis for the DCTRWs on the basis of Refs. [35, 39]. To find the asymptotic behaviour of the DCTRWs the densities $\lambda(x)$ and $\psi_a(t)$ are replaced by the scaled densities $\lambda_h(x) = \lambda(x/h)/h$, $h > 0$, and $\psi_{a,\tau}(t) = \psi_a(t/\tau)/\tau$, $\tau > 0$, respectively [35, 39]. The parameters h and τ can be considered as the characteristic step length and waiting time, respectively. The Fourier transform of λ_h is $\mathcal{F}\lambda_h(k) = \mathcal{F}\lambda(hk)$, and the Laplace transform of $\psi_{a,\tau}$ is $\mathcal{L}\psi_{a,\tau}(s) = \mathcal{L}\psi_a(\tau s)$. We suppose that the distribution of the jump length is symmetric, $\lambda(-x) = \lambda(x)$, and has the finite second moment with the variance equal to σ^2, then the asymptotic behaviour for $\mathcal{F}\lambda_h$ is $1 - \mathcal{F}\lambda_h(k) \sim h^2\sigma^2 k^2/2$ as $h \to 0$, $k \in \mathbb{R}$ [39]. For the density $\psi_{a,\tau}$ one has the asymptotics $1 - \mathcal{L}\psi_{a,\tau}(s) \sim 2\tau\theta s/a$ as $\tau \to 0$. The latter asymptotics are also valid in general case for $\psi_\tau(t) = \psi(t/\tau)/\tau$, if the density ψ has the finite expectation equal to $2\theta/a$ [39]. After substitution of the above asymptotics into the Montroll–Weiss formula (II.2), one can see that the only possible nontrivial limit is

$$\mathcal{F}\mathcal{L}\rho_{\mathrm{DRW},h,\tau}(k,s) \equiv \frac{1 - \mathcal{L}\psi_{a,\tau}(s)}{s\left[1 - \mathcal{F}\lambda_h(k)\mathcal{L}\psi_{a,\tau}(s)\right]}$$

$$\to \mathcal{F}\mathcal{L}\rho_{\mathrm{DE}}(k,s) \quad \text{as } h \to 0 \text{ and } \tau \to 0 \quad \text{(IV.1)}$$

under the scaling relation $h^2/\tau = 1$, where

$$\mathcal{F}\mathcal{L}\rho_{\mathrm{DE}}(k,s) = \frac{1}{s + a\sigma^2 k^2/(4\theta)}$$

The inverse Fourier–Laplace transform implies that the density ρ_{DE} is a solution of the diffusion equation (DE)

$$\frac{\partial \rho_{\mathrm{DE}}}{\partial t} - \frac{a\sigma^2}{4\theta}\frac{\partial^2 \rho_{\mathrm{DE}}}{\partial x^2} = 0 \qquad \text{(IV.2)}$$

with the initial condition

$$\rho_{\mathrm{DE}}|_{t=0} = \delta(x). \qquad \text{(IV.3)}$$

Thus, the asymptotic behaviour of the DCTRWs is given by the diffusion equation (IV.2). Note that the diffusion equation is obtained from the telegraph equation (III.4) by omitting the second time derivative.

Note also that $\mathcal{F}\mathcal{L}\rho_{\mathrm{TE}}(0,s) = 1/s$ and $\mathcal{F}\mathcal{L}\rho_{\mathrm{DE}}(0,s) = 1/s$, or, equivalently, $\mathcal{F}\rho_{\mathrm{TE}}(0,t) = 1$ and $\mathcal{F}\rho_{\mathrm{DE}}(0,t) = 1$ for $t \geq 0$, i. e., the law of conservation of particles is valid for both the telegraph and diffusion approximations.

The other possible limits are $\mathcal{F}\mathcal{L}\rho_{\mathrm{DRW},h,\tau}(k,s) \to 1/s$ as $h \to 0$, and $\mathcal{F}\mathcal{L}\rho_{\mathrm{DRW},h,\tau}(k,s) \to 1/s$ as $\tau \to 0$ and $h/\sqrt{\tau} \to 0$. The former is the limit of infinitesimal jumps with finite mean waiting time. This is the correct limit [to be] obtained in Refs. [28, 32]. These asymptotics are trivial, since the inverse Fourier–Laplace transform of $1/s$ gives $\delta(x)$, $t \geq 0$, which means that the particle does not leave an initial position.

The diffusion limit, under the scaling relation $h^2/\tau = $ const, is valid for a variety of symmetric (unbiased) random walks, both continuous-time and discrete-time ones [4, 25, 34, 38]. In particular, it is valid for the symmetric Bernoulli random walk, which is discrete both in time and space, when the waiting time is exactly τ and the jumps are exactly $\pm h$ with equal probabilities, i. e., the corresponding probability densities are

$$\psi_{\mathrm{d}}(t) = \delta(t - \tau), \quad \tau > 0, \quad \text{and}$$

$$\lambda_{\mathrm{d}}(x) = \frac{1}{2}[\delta(x - h) + \delta(x + h)], \quad h > 0. \quad \text{(IV.4)}$$

However, in contrast to the CTRWs, in which waiting time can be arbitrarily small and, hence, the particle moves with an infinite speed, the speed of the motion of the particle in the Bernoulli random walk is finite. Besides, it turns out that the solution of the telegraph equation approximates the density for the Bernoulli random walk better than that of the diffusion equation [40].

It is necessary to note here that the explanation, given in Ref. [40], does not allow us to judge whether the telegraph equation approximates the random walk better than the diffusion equation in a wide time interval? To check this conclusion we have performed calculations, in which the binomial distribution, being the distribution of the walking particle, was evaluated through the first two terms of Stirling's asymptotic series for the gamma function [41] $\Gamma(z) = \sqrt{2\pi/z}\,(z/\,e)^z\,[1+1/(12z)+O(|z|^{-2})]$ as $|z| \to \infty$, $z \in \mathbb{C}$, $|\arg z| \le \pi - \Delta$, $\Delta > 0$, which is more accurate than conventional Stirling's formula. These calculations have confirmed the conclusion up to time $t \approx 100\tau$, which seems to be asymptotic. Obviously, the solutions of both the diffusion equation and the telegraph one do not approximate the binomial distribution at short time.

The reason, by which the solution of the telegraph equation approximates the Bernoulli random walk better than that of the diffusion equation, seems to be clear: the Bernoulli particle moves with finite velocity. However, the probability density for the waiting time ψ_{d}, Eq. (IV.4), can be weakly approximated with arbitrary accuracy by the gamma distribution:

$$\psi_\mu(t) \equiv \frac{1}{\Gamma(\mu)}\frac{\mu}{\tau}\left(\frac{\mu t}{\tau}\right)^{\mu-1}\mathrm{e}^{-\mu t/\tau}$$

$$\xrightarrow{\text{weakly}} \delta(t-\tau) \quad \text{as } \mu \to +\infty, \quad \text{(IV.5)}$$

where Γ is the Gamma function; this can easily be derived with the Laplace transform. In the random walk with the distribution ψ_μ of the waiting time the particle moves with infinite velocity, since the waiting time can generally be arbitrarily small. However, for sufficiently large μ the telegraph approximation to this random walk is clearly better than the diffusion one, since $\psi_\mu(t)$ is very "close" to $\delta(t-\tau)$.

Thus, in spite of the diffusion asymptotic behaviour for the DCTRWs and the infinite speed of the motion of the particle, the question still remains: which of the telegraph or the diffusion equations gives better approximation to the DCTRWs?

V. COMPARISON OF THE TELEGRAPH AND DIFFUSION APPROXIMATIONS TO THE DCTRWS

To compare the telegraph and diffusion equations with the DCTRWs we consider two particular DCTRWs. One of them is a discrete-space random walk, the other is the continuous-space one.

The *discrete-space* random walk takes place on a regular one-dimensional lattice, where the distance between nearest-neighbour points is σ, and the coordinates of the lattice points are $x_l = l\sigma$, $l \in \mathbb{Z}$. Jumps are made to the nearest-neighbour points with equal probability. This random walk is described by the probability density

$$\lambda_{\mathrm{B}}(x) = \frac{1}{2}[\delta(x-\sigma) + \delta(x+\sigma)]. \quad \text{(V.1)}$$

The subscript $_{\mathrm{B}}$ means Bernoulli (since this is the same distribution as in the Bernoulli random walk). The variance of this distribution is equal to σ^2.

The *continuous-space* random walk has the Gaussian distribution of the jump length with the variance equal also to σ^2:

$$\lambda_{\mathrm{G}}(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\,\mathrm{e}^{-x^2/(2\sigma^2)}, \quad \text{(V.2)}$$

its Fourier transform is $\mathcal{F}\lambda_{\mathrm{G}}(k) = \mathrm{e}^{-\sigma^2 k^2/2}$. The Gaussian distribution of the jump length was used in Ref. [29].

The densities of the discrete- and continuous-space DCTRWs are given in Appendix A by Eqs. (A.1) and (A.2), respectively. The telegraph and diffusion approximations are given by Eqs. (A.5) and (A.8), respectively.

All numerical results are obtained with the parameters $\sigma = 1$ and $\theta = 1$.

Figs. 1 and 2 show (in Cartesian and logarithmic scales) the densities of the continuous- and discrete-space DCTRWs and their telegraph and diffusion approximations at intermediate values of time $t = 5$ and $t = 10$ with $a = 1$. Figs. 3 and 4 show differences between each of the two approximations and the densities of the continuous- and discrete-space DCTRWs, respectively, obtained for the same values of time and the parameters. Calculations show that in all the cases the L_2-discrepancy for the diffusion approximation is less than that for the telegraph one. In the case of the discrete-space random walk we calculated the l_2-discrepancy. Note also that the maximum absolute value of the difference for the diffusion approximation is in all the cases less than that for the telegraph approximation. Besides, it is important to emphasize that the telegraph approximation is incorrect with respect to the velocity of the motion of the particle, see Figs. 1 and 2 for $t = 5$. The telegraph model gives the finite velocity, while it is infinite in the DCTRWs. The diffusion model is correct in this respect, however the diffusion paradox remains.
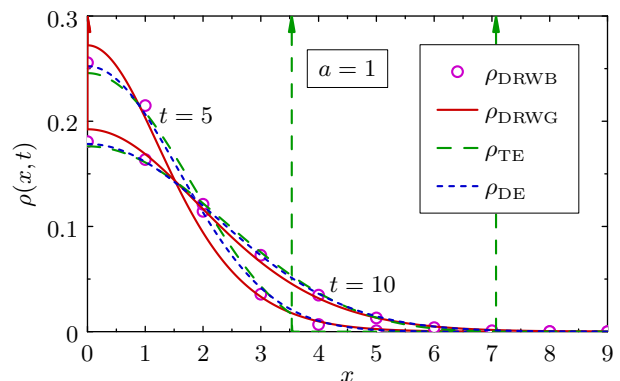

FIG. 1. (Color online) The densities $\rho_{\mathrm{DRWG}}(x,t)$ and $\rho_{\mathrm{DRWB}}(x_l,t)$ and the telegraph and diffusion approximations $\rho_{\mathrm{TE}}(x,t)$ and $\rho_{\mathrm{DE}}(x,t)$ at $t = 5$ and $t = 10$ with $\sigma = 1$, $a = 1$ and $\theta = 1$. The singular terms $\rho^{\mathrm{s}}_{\mathrm{DRWG}}$ and $\rho^{\mathrm{s}}_{\mathrm{TE}}$ are depicted by vertical arrows.
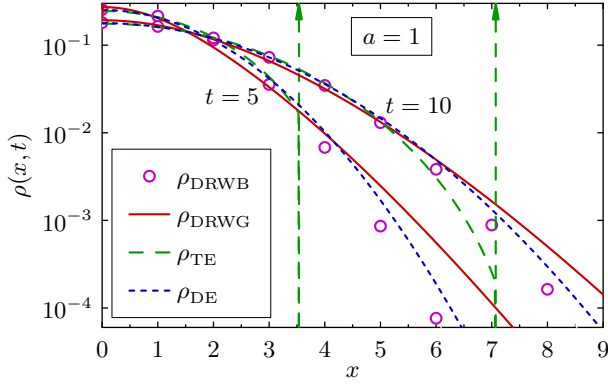
FIG. 2. (Color online) The same as in Fig. 1, with logarithmic scale on the vertical axis.

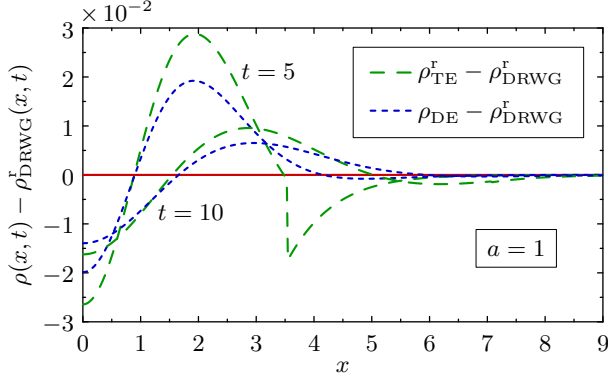FIG. 3. (Color online) The differences $\rho_{\mathrm{TE}}^{\mathrm{r}}(x,t) - \rho_{\mathrm{DRWG}}^{\mathrm{r}}(x,t)$ and $\rho_{\mathrm{DE}}(x,t) - \rho_{\mathrm{DRWG}}^{\mathrm{r}}(x,t)$ at $t = 5$ and $t = 10$ with $\sigma = 1$, $a = 1$ and $\theta = 1$. Note that the vertical scale unit is 10^{-2}.

FIG. 4. (Color online) The differences $\rho_{\mathrm{TE}}^{\mathrm{r}}(x_l,t) - \rho_{\mathrm{DRWB}}(x_l,t)$ and $\rho_{\mathrm{DE}}(x_l,t) - \rho_{\mathrm{DRWB}}(x_l,t)$ at $t = 5$ and $t = 10$ with $\sigma = 1$, $a = 1$ and $\theta = 1$. Note that the vertical scale unit is 10^{-2}.
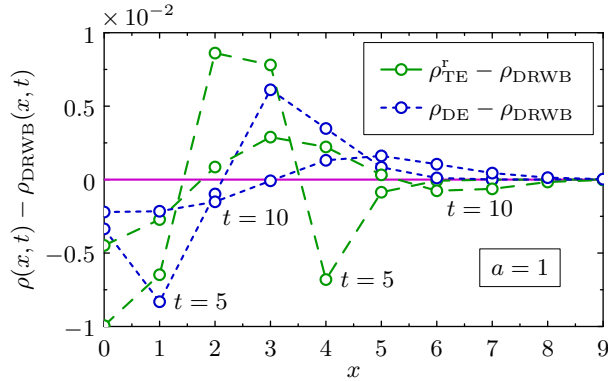
To assess the approximations at long times we have performed in Appendix B asymptotic comparison of the telegraph and diffusion approximations to the continuous-space DCTRW through their Fourier transforms.

Fig. 5 shows graphs resulting from the Fourier transforms $\mathcal{F}\rho_{\mathrm{DRWG}}(k,t)$, $\mathcal{F}\rho_{\mathrm{TE}}(k,t)$ and $\mathcal{F}\rho_{\mathrm{DE}}(k,t)$ at $t = 20$ and $t = 50$ with $a = 1$. These values correspond to the values of the small parameter $\varepsilon = 0.05$ and $\varepsilon = 0.02$, respectively (see Appendix B). Fig. 6 shows the integrands of the integrals (B.1) for the same values of time and parameters σ, a and θ as in Fig. 5. Note that the asymptotics (B.6) and (B.7) provide very good approximations (not shown in the figure) to the integrands.

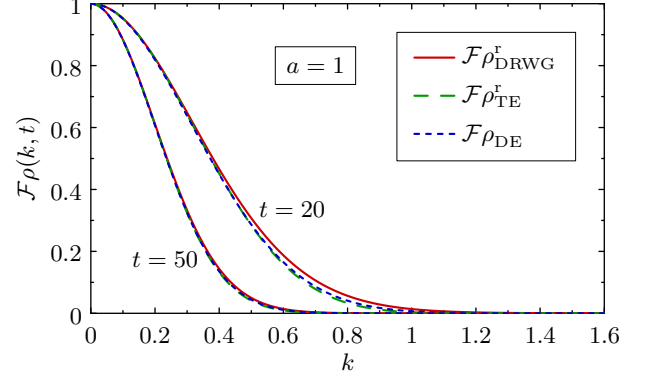FIG. 5. (Color online) Graphs resulting from the Fourier transforms $\mathcal{F}\rho_{\mathrm{DRWG}}^{\mathrm{r}}(k,t)$, $\mathcal{F}\rho_{\mathrm{TE}}^{\mathrm{r}}(k,t)$ and $\mathcal{F}\rho_{\mathrm{DE}}(k,t)$ at $t = 20$ and $t = 50$ with $\sigma = 1$, $a = 1$ and $\theta = 1$, i.e., $\varepsilon = 0.05$ and $\varepsilon = 0.02$.
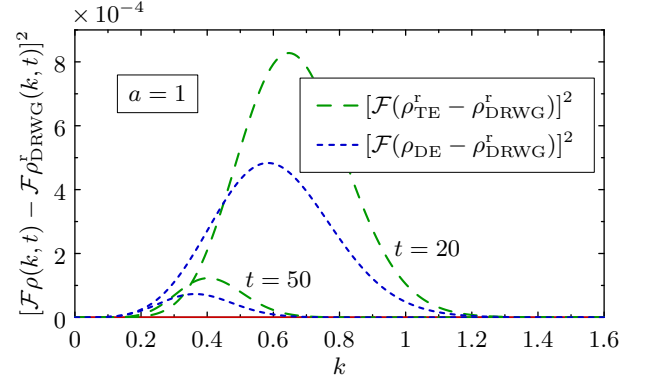

FIG. 6. (Color online) The integrands of the integrals (B.1) at $t = 20$ and $t = 50$ with $\sigma = 1$, $a = 1$ and $\theta = 1$, i.e., $\varepsilon = 0.05$ and $\varepsilon = 0.02$. Note that the vertical scale unit is 10^{-4}.

For comparison we show in Fig. 7 differences between each of the two approximations and the densities of the discrete-space DCTRW (not Fourier transforms), obtained at the same long times $t = 20$ and $t = 50$ with $a = 1$. Note that Fig. 7 is qualitatively similar to Fig. 4. In these cases the l_2-discrepancy for the diffusion approximation is also less than that for the telegraph one. We do not show differences between the two approximations and the densities of the continuous-space DCTRW for $t = 20$ and $t = 50$ because Fig. 6 illustrates this via the Fourier transforms.
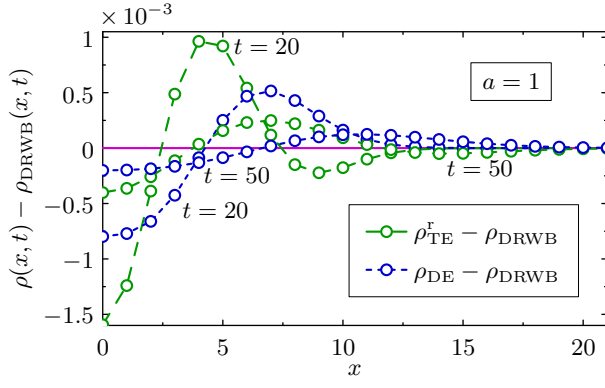
FIG. 7. (Color online) The differences $\rho_{\mathrm{TE}}^{\mathrm{r}}(x_l, t) - \rho_{\mathrm{DRWB}}(x_l, t)$ and $\rho_{\mathrm{DE}}(x_l, t) - \rho_{\mathrm{DRWB}}(x_l, t)$ at $t = 20$ and $t = 50$, $\sigma = 1$, $a = 1$ and $\theta = 1$. Note that the vertical scale unit is 10^{-3}. Cf. with Fig. 4.

Figs. 8 and 9 show the densities of the continuous- and discrete-space DCTRWs and their telegraph and diffusion approximations with the parameter $a = 0.1$. Fig. 8 corresponds to the small value of time $t = 20$ ($\varepsilon = 0.5$), which is equal to the mean waiting time. In this case the singular term $\rho_{\mathrm{DRWG}}^{\mathrm{s}}$ is quite "heavy": the probability that the particle stays at the initial point is $\Psi_{0.1}(20) \approx 0.37$. Note that the telegraph and diffusion "approximations" essentially differ from the densities of the DCTRWs. At the same time they are very close to each other. Fig. 9 corresponds to intermediate and long time: $t = 50$, $t = 100$ (intermediate) and $t = 500$ (long) ($\varepsilon = 0.2$, $\varepsilon = 0.1$ and $\varepsilon = 0.02$, respectively). For $t = 50$ the singular term $\rho_{\mathrm{DRWG}}^{\mathrm{s}}$ is still "heavy" enough: the probability that the particle stays at the initial point for the time $t = 50$ is $\Psi_{0.1}(50) \approx 0.08$. It is necessary to emphasize that all the results are in agreement with the asymptotics (B.8)–(B.10) (even for intermediate and small values of time): the differences between the two approximations are much less than the differences between each of them, and the densities of the DCTRWs, while the diffusion approximation is slightly better than the telegraph one.

The DCTRWs with small a can be explained as follows. Small values of a mean that the particle mostly remains in rest than jumps, since the mean waiting time is long enough. Small values of time can be described as lesser than or comparable with the mean waiting time $2\theta/a$, i. e., such that $at/\theta \lesssim 1$. Therefore, for small values of time both the diffusion and telegraph equations badly approximate the DCTRW, see Fig. 8. At the same time, if $at/\theta \sim 1$ then $t/\theta \gg 1$, i. e., the singular term (A.6) of the telegraph approximation is negligible. In this case the solution of the telegraph equation behaves like the solution of the diffusion equation. At long times the telegraph and diffusion approximations are almost undistinguishable and both are good, while the diffusion one is slightly better.
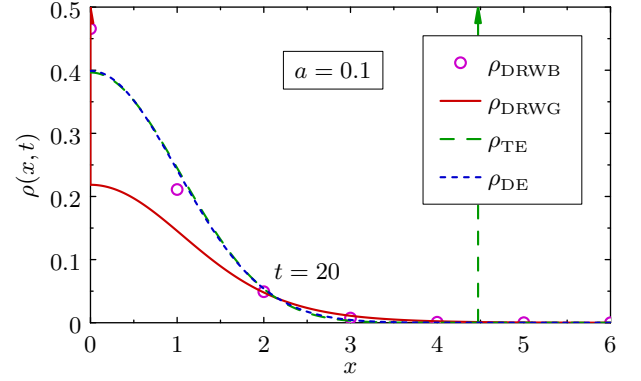

FIG. 8. (Color online) The densities $\rho_{\mathrm{DRWG}}(x, t)$ and $\rho_{\mathrm{DRWB}}(x_l, t)$ and the telegraph and diffusion approximations $\rho_{\mathrm{TE}}(x, t)$ and $\rho_{\mathrm{DE}}(x, t)$ at $t = 20$, $\sigma = 1$, $a = 0.1$ and $\theta = 1$ ($\varepsilon = 0.5$). The telegraph and diffusion approximations are very close to each other. The singular terms $\rho_{\mathrm{DRWG}}^{\mathrm{s}}$ and $\rho_{\mathrm{TE}}^{\mathrm{s}}$ are depicted by vertical arrows. Note that the singular term $\rho_{\mathrm{DRWG}}^{\mathrm{s}}$ is quite "heavy".
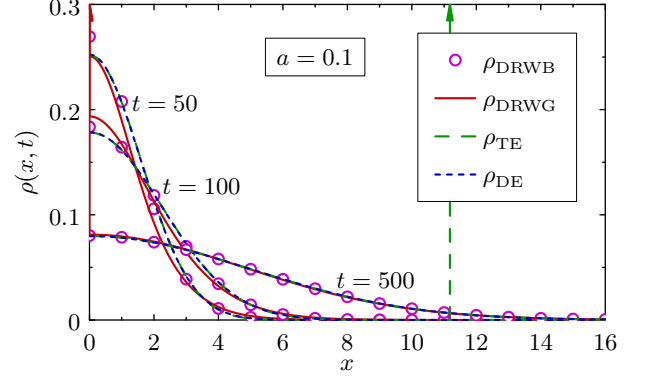

FIG. 9. (Color online) The densities $\rho_{\mathrm{DRWG}}(x, t)$ and $\rho_{\mathrm{DRWB}}(x_l, t)$ and the telegraph and diffusion approximations $\rho_{\mathrm{TE}}(x, t)$ and $\rho_{\mathrm{DE}}(x, t)$ at $t = 50$, $t = 100$ and $t = 500$, $\sigma = 1$, $a = 0.1$ and $\theta = 1$ ($\varepsilon = 0.2$, $\varepsilon = 0.1$ and $\varepsilon = 0.02$). The telegraph and diffusion approximations are almost undistinguishable in the figure. The singular terms $\rho_{\mathrm{DRWG}}^{\mathrm{s}}$ and $\rho_{\mathrm{TE}}^{\mathrm{s}}$ (only for $t = 50$) are depicted by vertical arrows. Cf. with Fig. 1.

VI. CONCLUSIVE REMARKS

We should remind that the telegraph equation describes transport phenomena with finite propagation speed, while the velocity of the motion of the particle in the DCTRWs is infinite, i. e., the DCTRWs do not simulate the most distinctive property of the telegraph equation. Moreover, asymptotic analysis and computations performed in this paper show that the diffusion equation gives better approximations to the DCTRWs than the telegraph equation. In other words, *the DCTRWs are always closer to their continuous limit than to the solutions of the telegraph equation*. This implies that, in contrast to the widespread opinion, the DCTRWs do not simulate

the telegraph equation at the microscopic level. Likewise, CTRWs with the exponentially distributed and general waiting time do not simulate the telegraph equation at the microscopic level.

Summing up, we can conclude that, first, the DCTRWs do not provide any correct microscopic interpretation of the one-dimensional telegraph equation, and second, the kinetic (exact) model of the telegraph equation is different from the model based on the DCTRWs.

An interesting question arises concerning discrete-space random walks with the distributions of the jump length λ_d, Eq. (IV.4), and the waiting time ψ_μ, Eq. (IV.5). For sufficiently large μ this walk is very close to the Bernoulli random walk, Eq. (IV.4). For $\mu = 1$ this is the discrete-space CTRW with the exponentially distributed waiting time $(1/\tau)\,e^{t/\tau}$. In the former case the telegraph approximation is better than the diffusion one, while in the latter case the diffusion approximation is better. Being incompletely stated yet, the problem is the following one: is there μ_0 such that for $\mu \in [1, \mu_0)$ the diffusion equation gives better approximation to these discrete-space CTRWs than the telegraph one, while for $\mu \in (\mu_0, \infty)$ the telegraph approximation is better?

Appendix A: Two particular (discrete- and continuous-space) DCTRWs and the telegraph and diffusion approximations

The density of the DCTRW with the distribution λ_B (V.1) of the jump length results from the discrete-space variant of the formula (II.3). We denote the density by ρ_{DRWB}. It is given by

$$\rho_{DRWB}(x_l, t) = \sum_{n=|l|}^{\infty} p_{a,n}(t)\lambda_{B,n}(x_l),$$

$$x_l = l\sigma, \quad l \in \mathbb{Z}, \quad (A.1)$$

where the probability density $\lambda_{B,n}$ is given by the scaled binomial distribution

$$\lambda_{B,n}(x_l)$$
$$= \begin{cases} \dfrac{n!}{2^n\,[(n+|l|)/2]!\,[(n-|l|)/2]!}, \\ \qquad\qquad |l| \le n \text{ and } n+l \text{ is even,} \\ 0, \qquad\qquad n < |l| \text{ or } n+l \text{ is odd,} \end{cases}$$

and $p_{a,n}$ stands for the probability p_n, corresponding to the distribution ψ_a. The Laplace transform of $p_{a,n}$ is

$$\mathcal{L}p_{a,n}(s) = \left(\frac{a}{\theta^2}\right)^n \frac{s + 2/\theta}{[(s + 1/\theta)^2 - (1-a)/\theta^2]^{n+1}}.$$

Therefore, the probability $p_{a,n}$ is given by

$$p_{a,n}(t) = \begin{cases} e^{-t/\theta}\left(\dfrac{a}{\theta^2}\right)^n\left[f'(t) + \dfrac{1}{\theta}f(t)\right], & 0 < a < 1, \\[2ex] e^{-t/\theta}\left[\dfrac{1}{(2n)!}\left(\dfrac{t}{\theta}\right)^{2n}\right. \\[2ex] \qquad\left. + \dfrac{1}{(2n+1)!}\left(\dfrac{t}{\theta}\right)^{2n+1}\right], & a = 1, \end{cases}$$

where

$$f(t) = \frac{1}{n!}\sum_{m=0}^{n} \frac{(n+m)!}{m!\,(n-m)!} \frac{t^{n-m}}{(2\sqrt{1-a}/\theta)^{n+m+1}}$$
$$\times \left[(-1)^m\,e^{\sqrt{1-a}\,t/\theta} + (-1)^{n+1}\,e^{-\sqrt{1-a}\,t/\theta}\right]$$

with corresponding limit as $a \to 1$.

The density of the DCTRW with the Gaussian distribution λ_G (V.2) of the jump length results from the formula (II.3). We denote the density by ρ_{DRWG}. It is given by

$$\rho_{DRWG}(x, t) = \rho^s_{DRWG}(x, t) + \rho^r_{DRWG}(x, t), \qquad (A.2)$$

where the singular term is

$$\rho^s_{DRWG}(x, t) = \Psi_a(t)\delta(x), \qquad (A.3)$$

with

$$\Psi_a(t) = \begin{cases} e^{-t/\theta}\left[\cosh\left(\dfrac{t}{\theta}\sqrt{1-a}\right)\right. \\[2ex] \qquad\left. + \dfrac{1}{\sqrt{1-a}}\sinh\left(\dfrac{t}{\theta}\sqrt{1-a}\right)\right], & 0 < a < 1, \\[2ex] e^{-t/\theta}\left(1 + \dfrac{t}{\theta}\right), & a = 1, \end{cases}$$
$$(A.4)$$

and the regular term is

$$\rho^r_{DRWG}(x, t) = \sum_{n=1}^{\infty} p_{a,n}(t)\frac{1}{\sqrt{2\pi n}\,\sigma}\,e^{-x^2/(2n\sigma^2)}.$$

The *telegraph approximation* is given by the solution to the initial-value problem (III.4), (III.5) for the telegraph equation [19, 42, 43]

$$\rho_{TE}(x, t) = \rho^s_{TE}(x, t) + \rho^r_{TE}(x, t), \qquad (A.5)$$

where the singular term is

$$\rho^s_{TE}(x, t) = e^{-t/\theta}\frac{1}{2}\left[\delta(x - vt) + \delta(x + vt)\right], \qquad (A.6)$$

and the regular term is

$$\rho_{\text{TE}}^{\text{r}}(x,t) = \text{e}^{-t/\theta}\,\frac{H(vt-|x|)}{2v\theta}\left[I_0\left(\frac{1}{\theta}\sqrt{t^2-\frac{x^2}{v^2}}\right)\right.$$

$$\left.+\,t\left(\sqrt{t^2-\frac{x^2}{v^2}}\right)^{-1}I_1\left(\frac{1}{\theta}\sqrt{t^2-\frac{x^2}{v^2}}\right)\right],\quad\text{(A.7)}$$

$v = \sqrt{a}\,\sigma/(\sqrt{2}\,\theta)$ is the velocity, $H(\cdot)$ is the Heaviside step function.

The *diffusion approximation* is given by the solution to the initial-value problem (IV.2), (IV.3) for the diffusion equation

$$\rho_{\text{DE}}(x,t) = \frac{\sqrt{\theta}}{\sqrt{\pi at}\,\sigma}\,\text{e}^{-\theta x^2/(a\sigma^2 t)}\,.\qquad\text{(A.8)}$$

Note that the significant difference between the density ρ_{DRWG} (A.2) of the DCTRW and the solution ρ_{TE} (A.5) of the telegraph equation is that the support of $\rho_{\text{DRWG}}^{\text{s}}$ (A.3) is localized at the starting point of the walk ($x = 0$), while the support of $\rho_{\text{TE}}^{\text{s}}$ (A.6) is localized at the moving front ($|x| = vt$). The term $\rho_{\text{DRWG}}^{\text{s}}$ is negligible for $at/\theta \gg 1$, while the term $\rho_{\text{TE}}^{\text{s}}$ is negligible for $t/\theta \gg 1$, the former condition being stronger. Note also that the support of $\rho_{\text{DRWG}}(\cdot,t)$ is $\mathbb{R}$, while the support of $\rho_{\text{TE}}(\cdot,t)$ is $[-vt, vt]$. However, the latter difference appears to be in general not so significant. Indeed, according to the DeMoivre–Laplace and the central limit theorems [25, 34] the solution (A.8) of the diffusion equation approximates the Bernoulli random walk (IV.4) with $h = \sqrt{a}\,\sigma$ and $\tau = 2\theta$, $t/(2\theta)$ in the expression (A.8) being the number of jumps. The solution (A.8) has the support $\mathbb{R}$, while the particle involved into the random walk has a finite velocity, nevertheless, the diffusion approximation is a classic and widely used one to the Bernoulli random walk.

Appendix B: Asymptotic comparison of the telegraph and diffusion approximations to the continuous-space DCTRW at long times

In this section the long time means $at/\theta \gg 1$, i.e., time is longer than the mean waiting time $2\theta/a$. We compare the telegraph and diffusion approximations to the continuous-space DCTRW at long times by evaluation the $L_2(\mathbb{R})$ norms of the differences $\rho_{\text{TE}}^{\text{r}}(\cdot,t) - \rho_{\text{DRWG}}^{\text{r}}(\cdot,t)$ and $\rho_{\text{DE}}(\cdot,t) - \rho_{\text{DRWG}}^{\text{r}}(\cdot,t)$. It is more convenient to consider the $L_2(\mathbb{R})$ norms of the Fourier transforms of the differences. Recall that the $L_2(\mathbb{R})$ norms of a function f and its Fourier transform $\mathcal{F}f$ are related by $\|f\|_{L_2(\mathbb{R})} = \sqrt{2\pi}\,\|\mathcal{F}f\|_{L_2(\mathbb{R})}$. Thus, we need to estimate the integrals

$$\int_0^\infty |\mathcal{F}\rho_{\text{TE}}^{\text{r}}(k,t) - \mathcal{F}\rho_{\text{DRWG}}^{\text{r}}(k,t)|^2\,\text{d}k\quad\text{and}$$

$$\int_0^\infty |\mathcal{F}\rho_{\text{DE}}(k,t) - \mathcal{F}\rho_{\text{DRWG}}^{\text{r}}(k,t)|^2\,\text{d}k\quad\text{(B.1)}$$

at long times (the integrals are taken over the interval $(0,\infty)$, rather than $(-\infty,\infty)$, since the integrands are even with respect to k).

The Fourier–Laplace transform of the density ρ_{DRWG} of the DCTRW with the Gaussian distribution λ_{G} of the jump length is the following one

$$\mathcal{F}\mathcal{L}\rho_{\text{DRWG}}(k,s)$$

$$= \frac{s+2/\theta}{(s+1/\theta)^2 - [1 - a(1 - \text{e}^{-\sigma^2 k^2/2})]/\theta^2}\,,$$

which results directly from the Montroll–Weiss formula (II.2) with $\mathcal{F}\lambda_{\text{G}}$ and $\mathcal{L}\psi_a$. This implies the Fourier transform

$$\mathcal{F}\rho_{\text{DRWG}}(k,t) = \text{e}^{-t/\theta}\left[\cosh\left(\frac{t}{\theta}\sqrt{1 - ar(k)}\right)\right.$$

$$\left.+\,\frac{1}{\sqrt{1-ar(k)}}\sinh\left(\frac{t}{\theta}\sqrt{1-ar(k)}\right)\right],$$

$$r(k) = 1 - \text{e}^{-\sigma^2 k^2/2},$$

the Fourier transform of the singular term being

$$\mathcal{F}\rho_{\text{DRWG}}^{\text{s}}(k,t) = \Psi_a(t),\qquad\text{(B.2)}$$

where Ψ_a is given by Eq. (A.4). Obviously $\mathcal{F}\rho_{\text{DRWG}}^{\text{r}}(k,t) \equiv \mathcal{F}\rho_{\text{DRWG}}(k,t) - \mathcal{F}\rho_{\text{DRWG}}^{\text{s}}(k,t) \to 0$ as $k \to \infty$.

The Fourier–Laplace transform (III.6) of the telegraph approximation can be rewritten as

$$\mathcal{F}\mathcal{L}\rho_{\text{TE}}(k,s) = \frac{s+2/\theta}{(s+1/\theta)^2 - (1 - a\sigma^2 k^2/2)/\theta^2}\,.$$

This implies the Fourier transform

$$\mathcal{F}\rho_{\text{TE}}(k,t) = \text{e}^{-t/\theta}\left[\cosh\left(\frac{t}{\theta}\sqrt{1 - a\frac{\sigma^2 k^2}{2}}\right)\right.$$

$$\left.+\left(\sqrt{1 - a\frac{\sigma^2 k^2}{2}}\right)^{-1}\sinh\left(\frac{t}{\theta}\sqrt{1 - a\frac{\sigma^2 k^2}{2}}\right)\right],$$

where we assume that if $a\sigma^2 k^2 > 2$, then $\sqrt{1 - a\sigma^2 k^2/2} = \text{i}\sqrt{a\sigma^2 k^2/2 - 1}$. The Fourier transform of the singular term is

$$\mathcal{F}\rho_{\text{TE}}^{\text{s}}(k,t) = \text{e}^{-t/\theta}\cos(vtk)$$

$$\equiv \text{e}^{-t/\theta}\cos\left(\frac{t}{\theta}\frac{\sqrt{a}\,\sigma k}{\sqrt{2}}\right).\quad\text{(B.3)}$$

Obviously $\mathcal{F}\rho_{\text{TE}}^{\text{r}}(k,t) \equiv \mathcal{F}\rho_{\text{TE}}(k,t) - \mathcal{F}\rho_{\text{TE}}^{\text{s}}(k,t) \to 0$ as $k \to \infty$.

The Fourier transform of the diffusion approximation is

$$\mathcal{F}\rho_{\text{DE}}(k,t) = \text{e}^{-(at/\theta)(\sigma^2 k^2/4)}\,.$$

The "singular terms" $\mathcal{F}\rho_{\mathrm{DRWG}}^{\mathrm{s}}$ (B.2) and $\mathcal{F}\rho_{\mathrm{TE}}^{\mathrm{s}}$ (B.3) are exponentially small as $t \to \infty$ and, therefore, negligible on any finite interval with respect to k for large t.

To estimate the integrals (B.1) at long times it is convenient to introduce the small parameter $\varepsilon = (at/\theta)^{-1} \ll 1$ and change the variable k for $\zeta = \sigma k/(2\sqrt{\varepsilon}) = (\sigma k/2)\sqrt{at/\theta}$. Then we have in the new variable the Fourier transform of the density of the continuous-space DCTRW

$$\mathcal{F}\rho_{\mathrm{DRWG}}(\zeta, t) = \mathrm{e}^{-1/a\varepsilon} \left[\cosh\left(\frac{\sqrt{1 - ar(\zeta)}}{a\varepsilon} \right) \right.$$
$$\left. + \frac{1}{\sqrt{1 - ar(\zeta)}} \sinh\left(\frac{\sqrt{1 - ar(\zeta)}}{a\varepsilon} \right) \right],$$

$$r(\zeta) = 1 - \mathrm{e}^{-2\varepsilon\zeta^2},$$

the "singular term" being $\mathcal{F}\rho_{\mathrm{DRWG}}^{\mathrm{s}}(\zeta, t) = \mathrm{e}^{-1/a\varepsilon}\left[\cosh\left(\sqrt{1-a}/a\varepsilon\right) + \sinh\left(\sqrt{1-a}/a\varepsilon\right)/\sqrt{1-a} \right]$, the Fourier transform of the telegraph approximation

$$\mathcal{F}\rho_{\mathrm{TE}}(\zeta, t) = \mathrm{e}^{-1/a\varepsilon} \left[\cosh\left(\frac{\sqrt{1 - 2a\varepsilon\zeta^2}}{a\varepsilon} \right) \right.$$
$$\left. + \frac{1}{\sqrt{1 - 2a\varepsilon\zeta^2}} \sinh\left(\frac{\sqrt{1 - 2a\varepsilon\zeta^2}}{a\varepsilon} \right) \right], \quad \text{(B.4)}$$

the "singular term" being $\mathcal{F}\rho_{\mathrm{TE}}^{\mathrm{s}}(\zeta, t) = \mathrm{e}^{-1/a\varepsilon} \cos\left(\sqrt{2/a\varepsilon}\,\zeta\right)$, and the Fourier transform of the diffusion approximation

$$\mathcal{F}\rho_{\mathrm{DE}}(\zeta, t) = \mathrm{e}^{-\zeta^2}.$$

The integrals (B.1), within a factor of $2/\sigma$, become in the new variable

$$\sqrt{\varepsilon} \int_0^\infty \left| \mathcal{F}\rho_{\mathrm{TE}}^{\mathrm{r}}(\zeta, t) - \mathcal{F}\rho_{\mathrm{DRWG}}^{\mathrm{r}}(\zeta, t) \right|^2 \mathrm{d}\zeta \quad \text{and}$$

$$\sqrt{\varepsilon} \int_0^\infty \left| \mathcal{F}\rho_{\mathrm{DE}}(\zeta, t) - \mathcal{F}\rho_{\mathrm{DRWG}}^{\mathrm{r}}(\zeta, t) \right|^2 \mathrm{d}\zeta. \quad \text{(B.5)}$$

The "regular term" $\mathcal{F}\rho_{\mathrm{DRWG}}^{\mathrm{r}}(\zeta, t) = \mathcal{F}\rho_{\mathrm{DRWG}}(\zeta, t) - \mathcal{F}\rho_{\mathrm{DRWG}}^{\mathrm{s}}(\zeta, t)$ tends to zero monotonously and exponentially as $\zeta \to \infty$, as well as the Fourier transform $\mathcal{F}\rho_{\mathrm{DE}}(\zeta, t)$. Therefore, the second integral of the integrals (B.5) can be approximated with arbitrary accuracy by the integral with the upper limit ζ_0 instead of ∞, if ζ_0 is sufficiently large.

For small ε the Fourier transform $\mathcal{F}\rho_{\mathrm{TE}}(\zeta, t)$ tends to zero monotonously and exponentially on the interval $[0, 1/\sqrt{2a\varepsilon}]$, where the square root in Eq. (B.4) is nonnegative. The "singular term" $\mathcal{F}\rho_{\mathrm{TE}}^{\mathrm{s}}$ can be neglected on this interval for small ε. Asymptotic behaviour of the "regular term" is $\mathcal{F}\rho_{\mathrm{TE}}^{\mathrm{r}}(\zeta, t) = \mathcal{F}\rho_{\mathrm{TE}}(\zeta, t) - \mathcal{F}\rho_{\mathrm{TE}}^{\mathrm{s}}(\zeta, t) = O(\zeta^{-1})$

as $\zeta \to \infty$. The reason for such a behaviour is that the regular term $\rho_{\mathrm{TE}}^{\mathrm{r}}(x, t)$ (A.7) is discontinuous at $|x| = vt$. Nevertheless, due to exponential decrease of $\mathcal{F}\rho_{\mathrm{TE}}^{\mathrm{r}}(\zeta, t)$ for small ε the first integral of the integrals (B.5) can be approximated with arbitrary accuracy by the integral with the upper limit ζ_0 instead of ∞, if ζ_0 is sufficiently large. Note that for any ζ_0 there exists sufficiently small ε such that $\zeta_0 < 1/\sqrt{2a\varepsilon}$, i.e., $[0, \zeta_0] \subset [0, 1/\sqrt{2a\varepsilon}]$.

Thus, for small ε the integrals (B.5) can be approximated with arbitrary accuracy by the integrals with the upper limit ζ_0 instead of ∞, where ζ_0 is sufficiently large.

The straightforward calculations imply uniform asymptotics on the interval $[0, \zeta_0]$:

$$\mathcal{F}\rho_{\mathrm{DRWG}}^{\mathrm{r}}(\zeta, t)$$
$$= \left\{ 1 + \left[\frac{a}{2}\left(\zeta^2 - \zeta^4 \right) + \zeta^4 \right] \varepsilon \right\} \mathrm{e}^{-\zeta^2} + O(\varepsilon^2)$$
$$\text{as} \quad \varepsilon \to 0$$

and

$$\mathcal{F}\rho_{\mathrm{TE}}^{\mathrm{r}}(\zeta, t) = \left\{ 1 + \left[\frac{a}{2}\left(\zeta^2 - \zeta^4 \right) \right] \varepsilon \right\} \mathrm{e}^{-\zeta^2} + O(\varepsilon^2)$$
$$\text{as} \quad \varepsilon \to 0$$

($\mathcal{F}\rho_{\mathrm{DRWG}}(\zeta, t)$ and $\mathcal{F}\rho_{\mathrm{TE}}(\zeta, t)$ have the same asymptotics). Therefore, the asymptotics for the differences are

$$\mathcal{F}\rho_{\mathrm{TE}}^{\mathrm{r}}(\zeta, t) - \mathcal{F}\rho_{\mathrm{DRWG}}^{\mathrm{r}}(\zeta, t)$$
$$= -\zeta^4 \, \mathrm{e}^{-\zeta^2} \, \varepsilon + O(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0 \quad \text{(B.6)}$$

and

$$\mathcal{F}\rho_{\mathrm{DE}}(\zeta, t) - \mathcal{F}\rho_{\mathrm{DRWG}}^{\mathrm{r}}(\zeta, t)$$
$$= -\left[\frac{a}{2}\left(\zeta^2 - \zeta^4 \right) + \zeta^4 \right] \mathrm{e}^{-\zeta^2} \, \varepsilon + O(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0.$$
$$\text{(B.7)}$$

Therefore, at long times (for small ε) the integrals (B.5) are

$$\sqrt{\varepsilon} \int_0^\infty \left| \mathcal{F}\rho_{\mathrm{TE}}^{\mathrm{r}}(\zeta, t) - \mathcal{F}\rho_{\mathrm{DRWG}}^{\mathrm{r}}(\zeta, t) \right|^2 \mathrm{d}\zeta$$

$$\simeq \varepsilon^{2.5} \int_0^\infty \zeta^8 \, \mathrm{e}^{-2\zeta^2} \, \mathrm{d}\zeta = \sqrt{\frac{\pi}{2}} \frac{7!!}{2^9} \varepsilon^{2.5}$$

$$\approx 0.257 \, \varepsilon^{2.5} \quad \text{(B.8)}$$

and

$$\sqrt{\varepsilon} \int_0^\infty |\mathcal{F}\rho_{\mathrm{DE}}(\zeta,t) - \mathcal{F}\rho_{\mathrm{DRWG}}^{\mathrm{r}}(\zeta,t)|^2 \, \mathrm{d}\zeta$$

$$\simeq \varepsilon^{2.5} \int_0^\infty \left[\frac{a}{2}\left(\zeta^2 - \zeta^4\right) + \zeta^4\right]^2 \mathrm{e}^{-2\zeta^2} \, \mathrm{d}\zeta$$

$$= \sqrt{\frac{\pi}{2}} \Bigg[\left(\frac{3!!}{2^7} - \frac{5!!}{2^8} + \frac{7!!}{2^{11}}\right) a^2$$

$$+ \left(\frac{5!!}{2^7} - \frac{7!!}{2^9}\right) a + \frac{7!!}{2^9}\Bigg] \varepsilon^{2.5}$$

$$\approx \left(0.020\, a^2 - 0.110\, a + 0.257\right) \varepsilon^{2.5}. \quad \text{(B.9)}$$

Note that at long times the same integral for the difference $\mathcal{F}\rho_{\mathrm{TE}}^{\mathrm{r}} - \mathcal{F}\rho_{\mathrm{DE}}$ is

$$\sqrt{\varepsilon} \int_0^\infty |\mathcal{F}\rho_{\mathrm{TE}}^{\mathrm{r}}(\zeta,t) - \mathcal{F}\rho_{\mathrm{DE}}(\zeta,t)|^2 \, \mathrm{d}\zeta$$

$$\simeq \varepsilon^{2.5} \int_0^\infty \left[\frac{a}{2}\left(\zeta^2 - \zeta^4\right)\right]^2 \mathrm{e}^{-2\zeta^2} \, \mathrm{d}\zeta \approx 0.020\, a^2\, \varepsilon^{2.5}. \tag{B.10}$$

The asymptotic estimates (B.8) and (B.9) show that at long times the diffusion approximation to the density ρ_{DRWG} of the DCTRW is better in the $L_2(\mathbb{R})$ norm, than the telegraph approximation. At the same time, for small a the telegraph approximation is almost as good as the diffusion one.

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